VISUALIZING COORDINATE ACCELERATION AND CHRISTOFFEL SYMBOLS

Marcel Ritter
Department of Computer Science
University of Innsbruck
A-6020 Innsbruck, Austria
marcel.ritter@student.uibk.ac.at

Werner Benger
Center for Computation and Technology
Louisiana State University
Baton Rouge, LA-70803, USA
werner@cct.lsu.edu

ABSTRACT

Acceleration is the derivative of the velocity and thus usually understood as the second derivative of the coordinate location of a moving object. Riemannian Geometry provides the mathematical framework on how to write equations in coordinate systems that are not inertial systems. Not only has general relativity benefited from this formalism but also practical applications such as numerical computations on curvilinear meshes. In this framework acceleration is described as the directional derivative of the velocity in the direction of the velocity. This mathematical operation is formulated in tensor algebra via the Christoffel symbols, a geometric object of rank three which is not a tensor by itself. In this article we present an approach on how to visualize these Christoffel symbols as indicator of the *coordinate-acceleration* and the curvature of space.

KEYWORDS

Coordinate-acceleration, tensor-field visualization, geodesics, space-time curvature, numerical relativity

1. INTRODUCTION

Acceleration in non-inertial coordinates requires consideration of non-vanishing Christoffel symbols, which describe the tangential transport and occur in the geodesic equation when computing the shortest or longest path between two points. These Christoffel symbols are similar to tensors of rank three, yet they are not tensorial objects itself. Visualization of tensors of rank two already is subject of current research [15], even rarer is work on visualization of tensors of higher rank [12]. To our knowledge, no work has yet been done on the direct visualization of Christoffel symbols.

This paper presents an approach to visualize the effect of the Christoffel symbols as they determine the path of geodesics in a curved space-time involving strong gravity, which was explored in [14]. Visualization of curved space-time via geodesics is a common approach, analytic space-times have been studied for instance by K. Hamilton [10], Ellis [6], Clark [5], Blaga [3], Fechting [7], Zahn [16] and Benger [1]. A work applicable to purely numerical data was done already in 1992 by Bryson [4]. Geodesics were also analyzed in numerical space-times in the 2D (axisymmetric) era [13]. A. Hamilton implemented a real time flight simulator for a charged black hole using a projective technique to compute the paths of geodesics [9].

2. MATHEMATICAL BACKGROUND

Let ***q(s)*** be a parameterized curve in a manifold $R \rightarrow M$. Then $\dot{q} = d/ds\, q$ is the *velocity* along this curve, an element of the tangential space at *q(s)*. The second derivative, involving tangential transport from different tangential spaces *q(s)* and *q(s+ds)*, is given by the directional derivative ∇ of the velocity $\dot{q}$ in the direction of the curve $\nabla_{\dot{q}}$, i.e.: $\ddot{q} = \nabla_{\dot{q}} \dot{q}$. A *geodesic* is a straight line, which is the result of accelerate-free motion according to Newton's first law of classical mechanics, requiring $\nabla_{\dot{q}} \dot{q} = 0$. Written in a coordinate system, the acceleration is given as

$$\ddot{q} = (\ddot{q}^{\lambda} + \Gamma^{\lambda}_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu})\partial_{\lambda} \quad (1) \qquad \text{with} \qquad \Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}) \quad (2)$$

Hereby, $g_{\mu\nu}$ are the components of the metric tensor in this coordinate system. Acceleration is not just the sum of the second derivative of coordinate locations $\ddot{q}^{\lambda}$, but involves also the so-called Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$. These are "correction" factors that allow computing the acceleration in non-inertial coordinate systems. Well-known examples are the so-called centrifugal and coriolis "forces", which actually are the Christoffel symbols written in a rotating (and thus non-inertial) coordinate system. The Christoffel symbols are not tensors – they may be all zero in one coordinate system, but non-zero in another coordinate system, which is not possible for a tensor. For a geodesic q with $\nabla_{\dot{q}}\dot{q} = 0$ it follows

$$\ddot{q}^{\lambda}\partial_{\lambda} = -\Gamma^{\lambda}_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu}\partial_{\lambda} \quad (3)$$

i.e. the Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$ applied to the velocity of a curve directly represent the second derivative $\ddot{q}^{\lambda}\partial_{\lambda}$ of the coordinate location, which can casually be seen as the acceleration as measured relative to this coordinate system – an observer in this coordinate system would misinterpret this measurement as the acceleration, when not knowing if he is inertial or not, which gives raise to the interpretation of the Christoffel symbols as mysterious virtual forces such as the centrifugal and coriolis force. Virtual forces have no physical cause but are merely due to a misinterpretation of a coordinate system as being inertial while it is it not. However, an inertial system does not necessarily exist at all. General relativity states that in the presence of masses there is no inertial system any more. Gravity itself is then considered as such a virtual force, ultimately described via the Christoffel symbols that are computed by the metric tensor field describing the curvature of space-time [11]. The term $\ddot{q}^{\lambda}\partial_{\lambda}$ is called ***coordinate-acceleration*** throughout the paper.

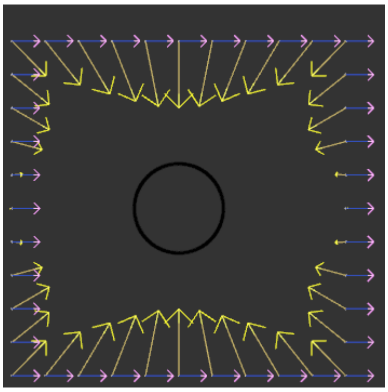

Figure 1: Illustration of the coordinate-acceleration (black/yellow arrows) on photons moving in positive x-direction (grey/blue arrows). The black circle indicates the event horizon of a Schwarzschild [11] black hole.

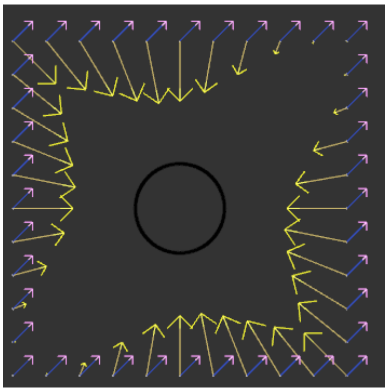

Figure 2: Illustration of the coordinate-acceleration (black/ yellow arrows) on photons moving in 45 degree rotated direction (grey/blue arrows) compared to figure (1).

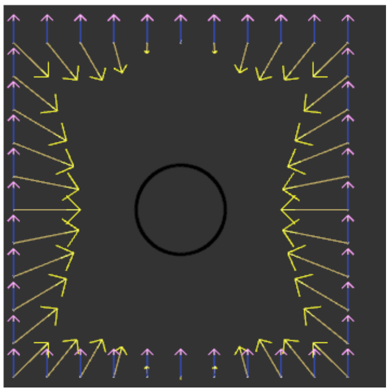

Figure 3: Like *figure (2)* but 90 degree rotation of the moving direct compared to *figure (1)*. The *coordinate-acceleration* is still pointing to the center of gravity.

3. VISUALIZING SPATIAL COORDINATE ACCELERATION

The *coordinate-acceleration* is a vector-like quantity dependent on the metric and the vector $\dot{q}^{\lambda}$ is the velocity of a photon. Both vector fields can be visualized using arrows, as illustrated in *figure (1),* based on the metric tensor field of a non-rotating (Schwarzschild) black hole. The different fields are shown in different grey levels. The Christoffel field maps the moving direction to the *coordinate-acceleration* following *eqn. (3)*. To investigate the properties of this mapping the direction vectors are rotated and again the *coordinate-acceleration* visualized, *figure (2)* and *(3)*.

Figure (1), *(2)* and *(3)* demonstrate that the directions of the *coordinate-acceleration* are always oriented towards the center of the black hole, but the magnitude changes. The magnitude decreases the closer the vector is located to the central axis through the black hole in direction of $\dot{q}^{\lambda}$. Evidently this is a non linear mapping due to the Christoffel symbols as geometric object of rank three. A mapping of this kind could not result from a linear transformation such as provided by a tensor field of rank two. Applying a matrix (which represents an object of rank two) to a vector would result in the output vector to be rotated same as the input vector rotates.

To enhance the visualization of *coordinate-acceleration* we utilize vector speckles [2], as shown in *figure (5)*. Colorization of the speckles is possible in addition to indicate direction and magnitude of the acceleration. Vector speckles are better readable compared to arrows especially when many samples are illustrated while still providing the same information content, see *figure (4)* and *(5)*. The *coordinate-acceleration* is pointing to the center of the Schwarzschild black hole. The camera was slightly tilted in *figure (5)* such that the vectors are not all perpendicular to the camera. Thus, the color and the shape of the speckles provide the directional information as the vector arrow icons.

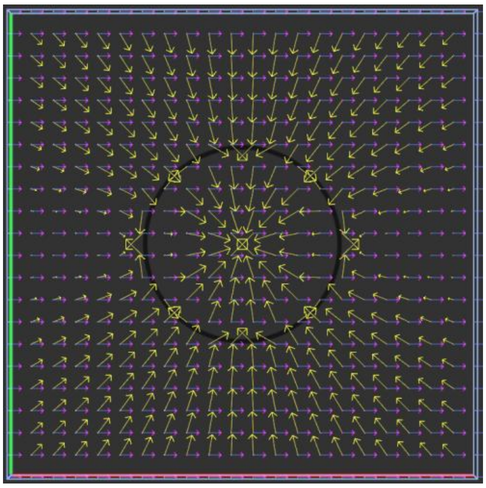

Figure 4: Grey/blue arrows illustrate the velocity and black/yellow arrows the coordinate-acceleration.

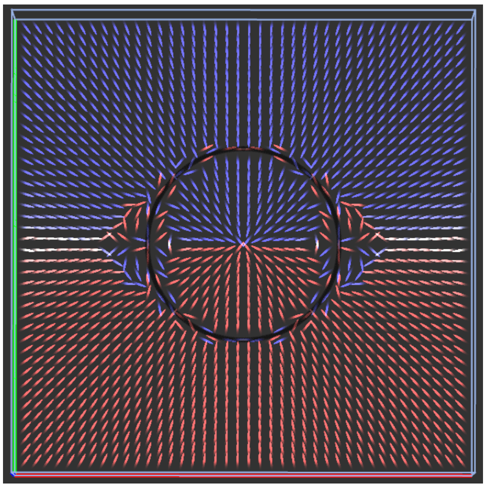

Figure 5: Vector speckles [2] are used to illustrate the coordinate-acceleration. The visualization of the velocity is omitted.

The technique transports over well to three dimensional data sets. *Figures (6)* and *(7)* show different spatial distributions in a three dimensional volume, visualizing a Kerr space-time [11] which describes a rotating black hole. Here, velocity is normalized and pointing in positive x-direction for all shown vector speckles. *Figure (6)* and *(7)* illustrate how the *coordinate-acceleration* field loses its spherical symmetry around the center with increased angular momentum. Certain oriented regions of the volume are emphasized by the random seeding approach. Note the bright (and red-colored), sparse looking region right of the center of *figure (7)*. Here all vector speckles point away from the direction of dominant geodesics within the Kerr space-time.

Figure (8) and *(9)* show geodesics seeded as bundles of small circles. The geodesics are computed using Dop853 integration [8] with initial conditions for origin and velocity. They are enhanced by showing the spatial *coordinate-acceleration* along the line and by coloring the line with a color map driven by the time dimension of the *coordinate-acceleration*. Photons are accelerated forward in time when moving towards the black hole. They are accelerated backwards in coordinate time in certain regions after passing it. The event horizon is shrinking with increasing angular momentum [11]. Geodesics are breaking at a closer distance to the center, *figure (9)*. Geodesics are symmetrical and spatial acceleration is pointing towards the center in *figure (8)*, in contrast to *figure (9)*. The bundles illustrate that geodesics are contracted when approaching the black hole close to the axis of the center of gravity. They are expanding when passing at farer distance. This property remains when introducing the angular momentum.

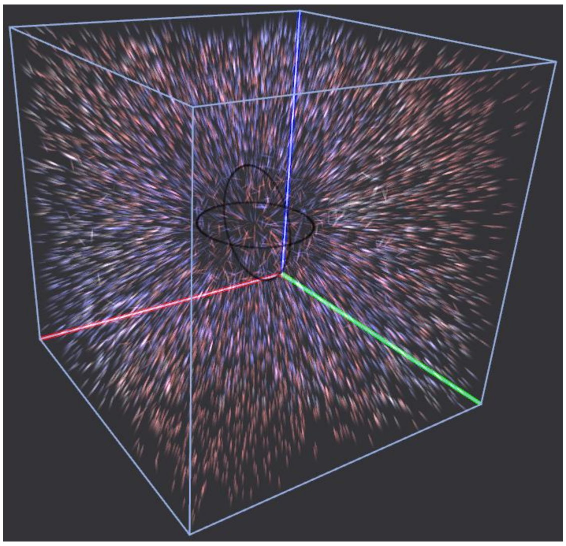

Figure 6: Illustration of the coordinate-acceleration sampled on a random point distribution within a 3D Kerr [11] space-time with no angular momentum, a=0, mass m=0.2.

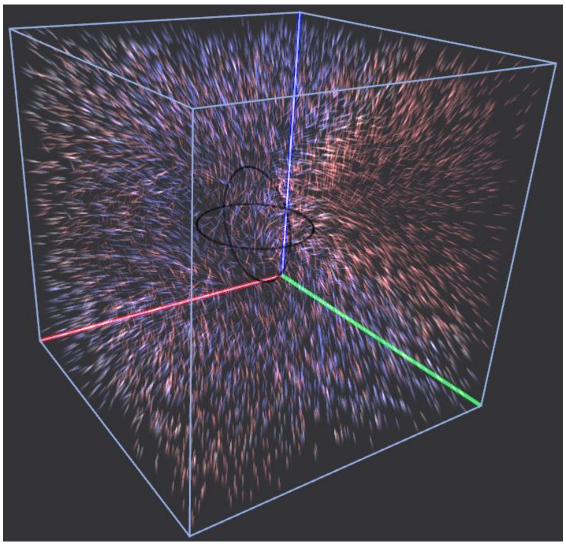

Figure 7: Like *figure (6)*. Kerr spacetime with an angular momentum of a=m=0.2, vector speckles of coordinate acceleration indicating non-spherical symmetry

Figures were created on an Intel Core Duo T8300@2.4Ghz system with 4GB RAM and Geforce 8400M GS Graphics card. Geodesic integration, such as *figure (8)*, takes about 30 seconds. One single integration step was measured with 0.0076 seconds in average. The result is fully interactive for camera navigation.

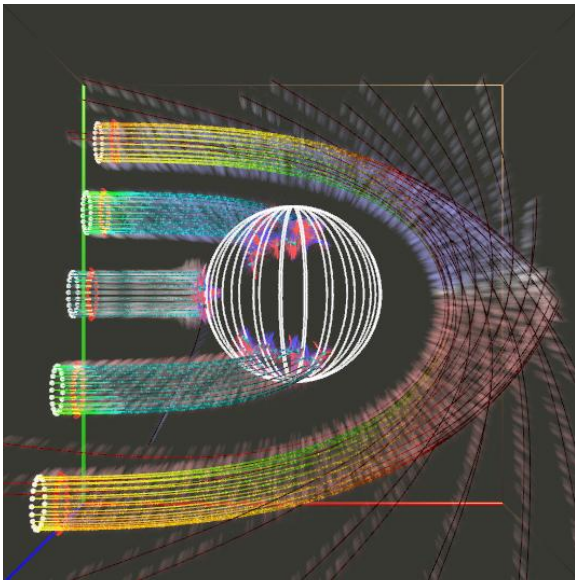

Figure 8: Visualization of enhanced geodesic tubes, (a=0). Geodesics are seeded on an array of circular shapes. Vector speckles and the color-map of the lines visualize the four dimensional coordinate-acceleration.

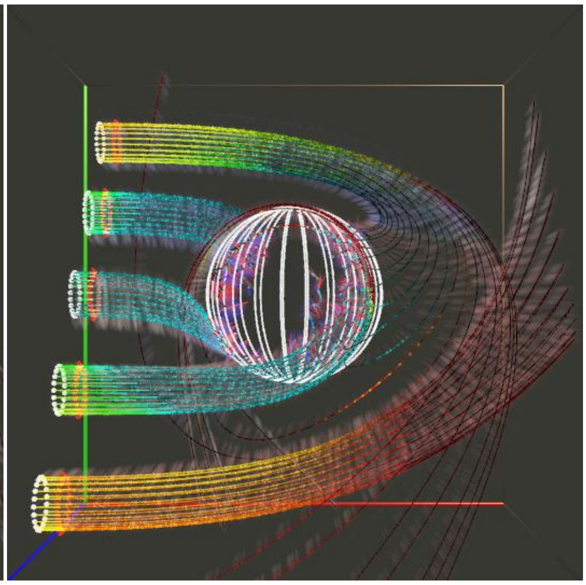

Figure 9: Like *figure (8)* but non-zero angular momentum a=0.2, which leads to asymmetries.

CONCLUSION

An interactive technique to visualize a geometric object of rank three, the Christoffel symbol, was demonstrated. It allows illustrating the four dimensional *coordinate-acceleration* by enhancing the three-dimensional visualization of geodesics. The technique has been exemplified via a numerically sampled Kerr space-time. The approach is directly applicable to numerical data sets, e.g. CFD data on curvilinear meshes or data stemming from numerical relativity research.

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